ADDITIONAL SYMMETRIES OF CONSTRAINED CKP AND BKP HIERARCHIES

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Abstract. The additional symmetries of the constrained CKP (cCKP) and BKP (cBKP) hierarchies are given by their actions on the Lax operators, and their actions on the eigenfunction and adjoint eigenfunction $\{\Phi_i, \Psi_i\}$ are presented explicitly. Furthermore, we show that acting on the space of the wave operator, ∂_k^* forms new centerless $W_{1+\infty}^{cC}$ and $W_{1+\infty}^{cB}$-subalgebra of centerless $W_{1+\infty}$ respectively. In order to define above symmetry flows ∂_k^* of the cCKP and cBKP hierarchies, two vital operators Y_k are introduced to revise the additional symmetry flows of the CKP and BKP hierarchies.

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1. Introduction

The research on Kadomtsev-Petviashvili (KP) hierarchy [1, 2] is one of the most important topics in the development of the theory of integrable systems. A specific interesting aspect on this topic is additional symmetry [3–8]. Additional symmetries are special symmetries which are not contained in the KP hierarchy and do not commute with each other. The additional symmetry flows of the KP hierarchy form an infinite dimensional algebra $W_{1+\infty}$ [5, 8]. More recently, there are several new results about partition function in the matrix models and Seiberg-Witten theory associated with additional symmetries, string equation and Virasoro constraints of the KP hierarchy [9–13]. In fact, there is a parallel research line on additional symmetries of evolution equations and their Lie algebraic structures, particularly for $1+1$ dimensional integrable equations [14], and the corresponding symmetries are called τ -symmetries and related Lax operators are constructed pretty systematically [15, 16].

It is well known that we can get different sub-hierarchies of the KP by different reduction conditions on Lax operator L. The first two important sub-hierarchies are CKP hierarchy [17] through a restriction $L^* = -L$ and BKP hierarchy [1] through a restriction $L^* = -\partial L \partial^{-1}$. So the additional symmetries of the CKP (or BKP) hierarchy have been constructed [18–21] and shown to be a new infinite dimensional algebra $W_{1+\infty}^C$ (or $W_{1+\infty}^B$), which is subalgebra of $W_{1+\infty}$. The third sub-hierarchy is constrained KP (cKP) hierarchy [22–24], which is obtained by setting a special form of Lax operator $L = \partial + \sum_{i=1}^m \Phi_i \partial^{-i} \Psi_i$, where Φ_i is the eigenfunction and Ψ_i is the adjoint eigenfunction of the cKP hierarchy. This is a relative new reduction condition and is different from the reduction conditions of previous two sub-hierarchies. Hence it is very natural to explore the additional symmetries of the cKP hierarchy. However, additional symmetry flows of the KP hierarchy break the special form of the Lax operator for the cKP hierarchy, and so it

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is highly non-trivial to find a suitable form of additional symmetry flow for this sub-hierarchy. To this end, it is well defined [25] by means of a crucial modification of the corresponding additional symmetry flows of the KP hierarchy. Note that a new and complicated operator X_k plays a very important role in this procedure.

Furthermore, the fourth sub-hierarchy and fifth sub-hierarchy are constrained CKP (cCKP) [26] and constrained BKP (cBKP) hierarchy [27], which are obtained by putting the combination of the previous mentioned reduction conditions, i.e., CKP and cKP conditions, BKP and cKP conditions. Some results associated cCKP and cBKP hierarchies have already been reported, for example, a Gramm-type τ-function [26] of the cCKP hierarchy and a Pfaffian-form τ-function [27] of the cBKP are also constructed from the vacuum solution, the dimensional reductions of the CKP and BKP hierarchies are presented in [28]. Moreover, determinant representations of the τ function, which are generated by the the gauge transformations, of the cCKP and cBKP hierarchies are obtained [29]. However, for our best knowledge, there is no any results on the additional symmetries of the cBKP and cCKP hierarchies. So, we shall fill the gap in this paper by constructing additional symmetries for the two new sub-hierarchies and their actions on eigenfunctions and adjoint eigenfunctions. The basic idea is to revise the additional symmetry flows of CKP and BKP hierarchies such that the new symmetry flows ∂_k^* will preserve the form of Lax operator of the cKP hierarchy. This will be realized by introducing two vital operators Y_k in sequel.

The paper is organized as follows. We first recall some basic results of the KP hierarchy, CKP hierarchy, BKP hierarchy and the constrained KP hierarchy in Section 2. The main results are stated and proved in Sections 3 and 4, which are the additional symmetries and its action on the eigenfunction and adjoint eigenfunction of the cCKP and cBKP hierarchies, and we further show that acting on the space of the wave operator, ∂_k^* forms new centerless $W_{1+\infty}^{cC}$ and $W_{1+\infty}^{cB}$-subalgebra of centerless $W_{1+\infty}$ respectively. Section 5 is devoted to conclusions and discussions.

2. BACKGROUND ON KP HIERARCHY

In this section we first give a brief introduction of KP hierarchy based on [1, 2]. Let the pseudo-differential operator

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + u_3\partial^{-3} + \cdots \tag{2.1}$$

be a Lax operator of the KP hierarchy, which is described by the associated Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L],\ n = 1, 2, 3, \cdots, \tag{2.2}$$

where $B_n = (L^n)_+ = \sum_{k=0}^{n} a_k\partial^k$ denotes the non-negative powers of ∂ in L^n, $\partial = \partial/\partial x$, $u_i = u_i(x = t_1, t_2, t_3, \cdots)$. The other notation $L_-^n = L^n - L_+^n$ will be used in the paper. The Lax operator L in eq.(2.1) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k\partial^{-k}$ in the following way

$$L = S\partial S^{-1}. \tag{2.3}$$

The dressing operator S satisfies Sato equation

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1, 2, 3, \cdots. \tag{2.4}$$

The wave function $w(t,z)$ of the KP hierarchy is defined by

$$w(t,z) = Se^{\xi(t,z)}, \tag{2.5}$$

where $\xi(t,z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots + t_n z^n \cdots$. The wave function satisfies the equations

$$L^k w(t,z) = z^k w(t,z), \frac{\partial}{\partial t_n} w = B_n w, k \in \mathbb{Z}, n \in \mathbb{N}. \tag{2.6}$$

Moreover $w(t,z)$ has a very simple expression by τ function of the KP hierarchy

$$w(t,z) = \frac{\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \cdots)}{\tau(t_1, t_2, t_3, \cdots)} e^{\xi(t,z)}. \tag{2.7}$$

Beside the existence of the Lax operator, wave function, τ function for the KP hierarchy, another important property is the Zakharov-Shabat equation and associated linear equation. In other words, KP hierarchy also has an alternative expression, i.e.,

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} + [B_m, B_n] = 0, \qquad m, n = 1, 2, 3, \cdots . \tag{2.8}$$

The eigenfunction $\Phi = \Phi(t_1, t_2, t_3, \cdots)$ and adjoint eigenfunction $\Psi = \Psi(t_1, t_2, t_3, \cdots)$ of KP hierarchy associated to (2.8) are defined by

$$\frac{\partial \Phi}{\partial t_n} = (B_n \Phi), \ \frac{\partial \Psi}{\partial t_n} = -(B_n^* \Psi), \tag{2.9}$$

where the symbol $*$ is the formal adjoint operation. For an arbitrary pseudo-differential operator $P = \sum_i p_i \partial^i$, $P^* = \sum_i (-1)^i \partial^i p_i$, and $(AB)^* = B^* A^*$ for pseudo-differential operators A and B. For example, $\partial^* = -\partial$, $(\partial^{-1})^* = -\partial^{-1}$.

To construct additional symmetries associated with the KP hierarchy, let us introduce Γ and Orlov-Shulman's M operator as $\Gamma = \sum_{i=1}^{\infty} i t_i \partial^{i-1}$ and $M = S\Gamma S^{-1}$. Dressing $[\partial_k - \partial^k, \Gamma] = 0$ gives $[\partial_k - B_k, M] = 0$, i.e.

$$\partial_k M = [B_k, M], \tag{2.10}$$

and then

$$\partial_k (M^m L^n) = [B_k, M^m L^n]. \tag{2.11}$$

Thus, the additional flows are defined by

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S, \tag{2.12}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L]. \tag{2.13}$$

Here $(M^m L^n)_-$ serves as the generator of the additional flows along the new time variables $t^*_{m,n}$. The additional flows $\partial^*_{m,n} = \dfrac{\partial}{\partial t^*_{m,n}}$ commute with the hierarchy, i.e. $[\partial^*_{m,n}, \partial_k] = 0$ but do not commute with each other, they indeed determine symmetries. Therefore these flows $\partial^*_{m,n}$ are also called additional symmetry flows, and they forms a centerless $W_{1+\infty}$ algebra [5, 8] acting on the spaces of the Lax operator L and wave operator S.

The CKP hierarchy is a reduction of the KP hierarchy through the constraint on L given by eq.(2.1) as

$$L^* = -L, \tag{2.14}$$

then L is called the Lax operator of the CKP hierarchy, and the associated Lax equation of the CKP hierarchy is

$$\frac{\partial L}{\partial t_n} = [B_n, L], n = 1, 3, 5, \cdots . \tag{2.15}$$

which compresses all even flows, i.e. the Lax equation of the CKP hierarchy has only odd flows. Additional symmetry flows [18] for the CKP hierarchy are defined as

$$\frac{\partial L}{\partial t^*_{m,l}} = -[(A_{m,l})_-, L], \tag{2.16}$$

where $A_{m,l}$ for the CKP hierarchy should satisfy

$$A^*_{m,l} = -A_{m,l}, \tag{2.17}$$

we can assume $A_{m,l}$ be

$$A_{m,l} = M^m L^l - (-1)^l L^l M^m, \tag{2.18}$$

where

$$M = S(\sum_{i=0}^{\infty}(2i+1)t_{2i+1}\partial^{2i})S^{-1}. \tag{2.19}$$

Acting on the space of the wave operator S, $\partial^*_{m,l}$ of the CKP hierarchy forms a new centerless $W^C_{1+\infty}$- subalgebra of centerless $W_{1+\infty}$.

If Lax operator L given by eq.(2.1) satisfies

$$L^* = -\partial L \partial^{-1}, \tag{2.20}$$

then L is called the Lax operator of the BKP hierarchy, the Lax equation of the BKP hierarchy also has only odd flows

$$\frac{\partial L}{\partial t_n} = [B_n, L], n = 1, 3, 5, \cdots . \tag{2.21}$$

Additional symmetry flows [19, 20] for the BKP hierarchy are given by

$$\frac{\partial L}{\partial t^*_{m,l}} = -[(A_{m,l})_-, L], \tag{2.22}$$

where $A_{m,l}$ for the BKP hierarchy should satisfy the constraint equation

$$A^*_{m,l} = -\partial A_{m,l} \partial^{-1}, \tag{2.23}$$

then $A_{m,l}$ for the BKP hierarchy could be chosen as

$$A_{m,l} = M^m L^l - (-1)^l L^{l-1} M^m L, \tag{2.24}$$

where

$$M = S(\sum_{i=0}^{\infty}(2i+1)t_{2i+1}\partial^{2i})S^{-1}. \tag{2.25}$$

Acting on the space of the wave operator S, $\partial^*_{m,l}$ of the BKP hierarchy forms another new centerless $W^B_{1+\infty}$ - subalgebra of centerless $W_{1+\infty}$.

We now turn to the cKP hierarchy [22–24]. The Lax operator L of the constrained KP hierarchy is given by

$$L=\partial+\sum_{i=1}^{m}\Phi_i\partial^{-i}\Psi_i, \tag{2.26}$$

where Φ_i (Ψ_i) is the (adjoint) eigenfunctions of this hierarchy, and the corresponding Lax equation is formulated as

$$\frac{\partial L}{\partial t_n}=[B_n,L], n=1,2,3,\cdots. \tag{2.27}$$

In particular, we stress that eigenfunctions and adjoint eigenfunctions $\{\Phi_i,\Psi_i\}$ are important dynamical variables in cKP hierarchy, so it is necessary to find the action of the correct additional symmetry flows on them. Here the correct flow ∂_τ means its action on L of the cKP hierarchy should be the form of [25]

$$(\partial_\tau L)_-=\sum_{i=1}^{m}(\tilde{A}\partial^{-1}\Psi_i+\Phi_i\partial^{-1}\tilde{B}), \tag{2.28}$$

which will result in its action on ($\partial_\tau\Phi_i=\tilde{A}$) and ($\partial_\tau\Psi_i=\tilde{B}$). However, in general, the additional symmetry flows of the KP hierarchy acting on L of the cKP hierarchy are not the form of eq.(2.28). In other words, these flows do not preserve the form of the Lax operator of the cKP hierarchy, the fact shows additional symmetry of the KP hierarchy is not consistent with cKP reduction condition automatically. Therefore, the additional symmetry flows have to be revised according to the analysis above, and then the correct additional symmetry flows of the cKP hierarchy are given by [25]

$$\partial_k^*L=[-(ML^k)_-+X_{k-1},L], k=0,1,2,3,\cdots, \tag{2.29}$$

where

$$X_{k-1}=0, k=0,1,2$$

and

$$X_{k-1}=\sum_{i=1}^{m}\sum_{j=0}^{k-2}(j-\frac{1}{2}(k-2))L^{k-2-j}(\Phi_i)\partial^{-1}(L^*)^j(\Psi_i), k\geq 3. \tag{2.30}$$

Furthermore [25],

$$\partial_k^*\Phi_i=\frac{k}{2}L^{k-1}(\Phi_i)+X_{k-1}(\Phi_i)+(A_{1,k})_+(\Phi_i), \tag{2.31}$$

$$\partial_k^*\Psi_i=\frac{k}{2}L^{*(k-1)}(\Psi_i)-X_{k-1}^*(\Psi_i)-(A_{1,k})_+^*(\Psi_i). \tag{2.32}$$

3. Additional symmetries of constrained CKP hierarchy

Let us consider the constrained CKP hierarchy, which is the C-type sub-hierarchy of cKP hierarchy. The Lax operator L of the cCKP hierarchy [26] is given by

$$L=\partial+\sum_{i=1}^{m}(q_i\partial^{-1}r_i+r_i\partial^{-1}q_i), \tag{3.1}$$

where q_i and r_i are eigenfunctions. The corresponding Lax equation of the cCKP hierarchy is defined by

$$\frac{\partial L}{\partial t_n} = [B_n, L], n = 1, 3, 5, \cdots . \tag{3.2}$$

For $k = 1, 3, 5, \cdots$, we first try to calculate the original additional symmetry flows of the CKP hierarchy as

$$\frac{\partial L}{\partial t^*_{1,k}} = -[(A_{1,k})_-, L], \tag{3.3}$$

where $A_{1,k} = ML^k - (-1)^k L^k M$. Thus

$$(\frac{\partial L}{\partial t^*_{1,k}})_- = [(A_{1,k})_+, L]_- + 2(L^k)_-. \tag{3.4}$$

In order to get its action on q_i and r_i, we need the form of $(L^k)_-$.

Lemma 3.1. The Lax operator L of the constrained CKP hierarchy given by eq.(3.1) satisfies the relation

$$(L^k)_- = \sum_{i=1}^{m}\sum_{j=0}^{k-1}(L^{k-j-1}(q_i)\partial^{-1}(L^*)^j(r_i) + L^{k-j-1}(r_i)\partial^{-1}(L^*)^j(q_i)), k = 1, 3, 5, \cdots \tag{3.5}$$

where

$$L(q_i) = L_+(q_i) + \sum_{j=1}^{m}(q_j\partial_x^{-1}(r_jq_i) + r_j\partial_x^{-1}(q_jq_i)).$$

However, $(L^k)_-$ is not in the form of

$$(\partial_\tau L)_- = \sum_{i=1}^{m}((\partial_\tau q_i)\partial^{-1}r_i + (\partial_\tau r_i)\partial^{-1}q_i + q_i\partial^{-1}(\partial_\tau r_i) + r_i\partial^{-1}(\partial_\tau q_i)), \tag{3.6}$$

as we expected for a correct flows. Here correctness means we can get the actions on $\partial_\tau q_i$ and $\partial_\tau r_i$ from eq.(3.6). This shows that eq.(3.4) can not imply its action on eigenfunctions $\partial^*_{1,k}q_i$ and $\partial^*_{1,k}r_i$.

Therefore, in order to get correct additional flows of the cCKP hierarchy, we revise eq.(3.3) and then define new flows by

$$\partial^*_k L = [-(A_{1,k})_- + Y_k, L], k = 1, 3, 5, \cdots , \tag{3.7}$$

where $A_{1,k} = ML^k - (-1)^k L^k M$ and Y_k is introduced such that the left hand side of eq.(3.7) will be the form of eq.(3.6). Next, we shall prove new flows ∂^*_k are correct additional symmetry flows of the cCKP hierarchy. First of all, Y_k is discussed, which is crucial to our purpose.

Lemma 3.2. For the cCKP hierarchy, the CKP reduction condition infers a constraint on Y_k,

$$Y^*_k = -Y_k, \ k = 1, 3, 5, \cdots . \tag{3.8}$$

Proof. The action of the additional flows ∂^*_k on the adjoint Lax operator L^* of the constrained CKP hierarchy can be obtained by two different ways. We first computet it by a formal adjoint operation on both sides of eq. (3.7), i.e.

$$\partial^*_k L^* = ([-(A_{1,k})_- + Y_k, L])^* = [L^*, -(A_{1,k})^*_- + Y^*_k], k = 1, 3, 5, \cdots . \tag{3.9}$$

The another way is to do a derivative with respect to t_k^* on L^* and use CKP reduction condition $L^*=-L$, then

$$\partial_k^* L^* = -\partial_k^* L = -[-(A_{1,k})_- + Y_k, L] = [L^*, (A_{1,k})_- - Y_k], k = 1, 3, 5, \cdots . \tag{3.10}$$

Comparing eq.(3.9) and eq.(3.10), we have

$$Y_k^* = -Y_k, \ \ k = 1, 3, 5, \cdots ,$$

with the help of $A_{1,k}^* = -A_{1,k}$. □

Thus we set Y_k as

$$Y_1 = 0, \tag{3.11}$$

$$Y_k = \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(L^{k-2-j}(q_i)\partial^{-1}(L^*)^j(r_i) + L^{k-2-j}(r_i)\partial^{-1}(L^*)^j(q_i)), k \geq 3, \tag{3.12}$$

and shall show Y_k satisfy the constraint given by eq.(3.8) .

Lemma 3.3.

$$Y_k = -Y_k^*. \tag{3.13}$$

Proof. $Y_1^* = Y_1 = 0$ is obvious, for $k = 3, 5, 7, \cdots$, we have

$$\begin{aligned}
Y_k^* &= \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(-(L^*)^j(r_i)\partial^{-1}L^{k-2-j}(q_i) - (L^*)^j(q_i)\partial^{-1}L^{k-2-j}(r_i))\\
&= \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(-(-1)^jL^j(r_i)\partial^{-1}(-1)^{k-2-j}(L^*)^{k-2-j}(q_i)\\
&- (-1)^jL^j(q_i)\partial^{-1}(-1)^{k-2-j}(L^*)^{k-2-j}(r_i))\\
&= \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(L^j(r_i)\partial^{-1}(L^*)^{k-2-j}(q_i) + L^j(q_i)\partial^{-1}(L^*)^{k-2-j}(r_i))\\
&= -\sum_{i=1}^{m}\sum_{l=0}^{k-2}(2l-(k-2))(L^{k-2-l}(q_i)\partial^{-1}(L^*)^l(r_i) + L^{k-2-l}(r_i)\partial^{-1}(L^*)^l(q_i))\\
&= -Y_k
\end{aligned}$$

In the fourth step we let $l = k-2-j$. □

Furthermore, in order to calculate $[Y_k, L]$ in the eq.(3.7), the following lemma is necessary.

Lemma 3.4. For the Lax operator L of the cCKP hierarchy in eq.(3.1) and a pseudo-different operator $X = \sum_{k=1}^{l} M_k\partial^{-1}N_k$, the equation

$$\begin{aligned}
[X, L]_- &= \sum_{k=1}^{l}(-L(M_k)\partial^{-1}N_k + M_k\partial^{-1}L^*(N_k))\\
&+ \sum_{i=1}^{m}(X(q_i)\partial^{-1}r_i + X(r_i)\partial^{-1}q_i - q_i\partial^{-1}X^*(r_i) - r_i\partial^{-1}X^*(q_i))
\end{aligned}$$

holds.

Based on the above lemma, we have

$$
\begin{aligned}
[Y_k, L]_- = -2(L^k)_- &+ k\sum_{i=1}^{m}(L^{k-1}(q_i)\partial^{-1}r_i + L^{k-1}(r_i)\partial^{-1}q_i)\\
&+k\sum_{i=1}^{m}(q_i\partial^{-1}L^{*(k-1)}(r_i) + r_i\partial^{-1}L^{*(k-1)}(q_i))\\
&+\sum_{i=1}^{m}(Y_k(q_i)\partial^{-1}r_i + Y_k(r_i)\partial^{-1}q_i)\\
&-\sum_{i=1}^{m}(q_i\partial^{-1}Y_k^*(r_i) + r_i\partial^{-1}Y_k^*(q_i)).
\end{aligned}
\tag{3.14}
$$

We are now in a position to calculate the explicit form of the right hand side of the new additional flow give by eq.(3.7).

Theorem 3.1. The additional flows acting on the eigenfunction q_i and r_i of the constrained CKP hierarchy are

$$\partial_k^* q_i = kL^{k-1}(q_i) + Y_k(q_i) + (A_{1,k})_+(q_i), k = 1, 3, 5, \cdots \tag{3.15}$$

and

$$\partial_k^* r_i = kL^{k-1}(r_i) + Y_k(r_i) + (A_{1,k})_+(r_i), k = 1, 3, 5, \cdots . \tag{3.16}$$

Proof. From the revised definition of the additional symmetry flows in eq.(3.7) of the cCKP hierarchy, by a short calculation, we have

$$\partial_k^* L_- = [-(A_{1,k})_- + Y_k, L]_- = [(A_{1,k})_+, L]_- + 2(L^k)_- + [Y_k, L]_-.$$

Using eq.(3.14) and the technical identity $[K, f\partial^{-1}g]_- = K(f)\partial^{-1}g - f\partial^{-1}K^*(g)$ (where f, g are arbitrary functions and K is a purely differential operator), note that $k = 1, 3, 5, \cdots$, it is followed by

$$
\begin{aligned}
\partial_k^* L_- = \sum_{i=1}^{m}(&(kL^{k-1}(q_i) + Y_k(q_i) + (A_{1,k})_+(q_i))\partial^{-1}r_i\\
&+ (kL^{k-1}(r_i) + Y_k(r_i) + (A_{1,k})_+(r_i))\partial^{-1}q_i\\
&+ q_i\partial^{-1}(k(L^*)^{k-1}(r_i) - Y_k^*(r_i) - (A_{1,k})_+^*(r_i))\\
&+ r_i\partial^{-1}(k(L^*)^{k-1}(q_i) - Y_k^*(q_i) - (A_{1,k})_+^*(q_i))
\end{aligned}
$$

thus

$$\partial_k^* q_i = kL^{k-1}(q_i) + Y_k(q_i) + (A_{1,k})_+(q_i) = k(L^*)^{k-1}(q_i) - Y_k^*(q_i) - (A_{1,k})_+^*(q_i),$$

$\partial_k^* r_i$ is also obtained at the same time. □

Corollary 3.1. The additional flows act on wave operator S of the constrained CKP hierarchy as

$$\partial_k^* S = (-(A_{1,k})_- + Y_k)S, k = 1, 3, 5, \cdots . \tag{3.17}$$

Theorem 3.2. The additional flows ∂_k^* commute with the constrained CKP hierarchy flows $\partial_{t_{2n+1}} = \dfrac{\partial}{\partial t_{2n+1}}$, i.e.

$$[\partial_k^*, \partial_{t_{2n+1}}] = 0. \tag{3.18}$$

Thus $\partial_k^*(k=1,3,5,\cdots)$ are indeed the additional symmetry flows of the cCKP hierarchy.

Proof. The proof starts with the definition

$$[\partial_k^*, \partial_{t_{2n+1}}]S = \partial_k^*(\partial_{t_{2n+1}}S) - \partial_{t_{2n+1}}(\partial_k^* S),$$

and using the action of the additional flows, we get

$$\begin{aligned}
[\partial_k^*, \partial_{t_{2n+1}}]S &= -\partial_k^*\left(L_-^{2n+1}S\right) + \partial_{t_{2n+1}}\left(((A_{1,k})_- - Y_k)S\right) \\
&= -(\partial_k^* L^{2n+1})_- S - (L^{2n+1})_-(\partial_k^* S) + \partial_{t_{2n+1}}((A_{1,k})_- - Y_k)S \\
&+ ((A_{1,k})_- - Y_k)(\partial_{t_{2n+1}}S).
\end{aligned}$$

Taking eq.(3.17) of Corollary 1 into the above formula, it is not difficult to compute

$$\begin{aligned}
[\partial_k^*, \partial_{t_{2n+1}}]S &= [(A_{1,k})_- - Y_k, L^{2n+1}]_- S + (L^{2n+1})_-((A_{1,k})_- - Y_k)S \\
&+ [(L^{2n+1})_+, (A_{1,k})_- - Y_k]_- S - ((A_{1,k})_- - Y_k)(L^{2n+1})_- S \\
&= [((A_{1,k})_- - Y_k), L^{2n+1}]_- S - [(A_{1,k})_- - Y_k, L_+^{2n+1}]_- S + [L_-^{2n+1}, (A_{1,k})_- - Y_k]S \\
&= [(A_{1,k})_- - Y_k, L_-^{2n+1}]_- S + [L_-^{2n+1}, (A_{1,k})_- - Y_k]S = 0
\end{aligned}$$

We have used the fact that $[L_+^{2n+1}, (A_{m,l}) - Y_k]_- = [L_+^{2n+1}, ((A_{1,k})_- - Y_k)]_-$ in the second step of the above derivation, since $(Y_k)_- = Y_k$ and $[L_+^{2n+1}, (A_{m,l})_+]_- = 0$. The last equality holds by virtue of $(P_-)_- = P_-$ for arbitrary pseduo-differential operator P. $\square$

Taking into account of $Y_k^* = -Y_k,\ k = 1,3,5,\cdots$, we can give the next theorem by a straightforward and tedious calculation.

Theorem 3.3. Acting on the space of the wave operator S of the constrained CKP hierarchy, ∂_k^* forms a new centerless $W_{1+\infty}^{cC}$-subalgebra of centerless $W_{1+\infty}$.

4. Additional symmetries of constrained BKP hierarchy

We now turn to the case of the constrained BKP hierarchy, this follows in a similar way of the case of the constrained CKP hierarchy. The Lax operator L of the constrained BKP hierarchy [27] is given by

$$L = \partial + \sum_{i=1}^{m}(q_i\partial^{-1}r_{i,x} - r_i\partial^{-1}q_{i,x}), \tag{4.1}$$

where the q_i and r_i are eigenfunctions. The corresponding Lax equation of the constrained BKP hierarchy is defined by

$$\frac{\partial L}{\partial t_n} = [B_n, L], n = 1,3,5,\cdots. \tag{4.2}$$

For $k = 1,3,5,\cdots$, at first we calculate the original additional symmetry flows of the BKP hierarchy as

$$\frac{\partial L}{\partial t_{1,k}^*} = -[(A_{1,k})_-, L], \tag{4.3}$$

where $A_{1,k} = M^m L^l - (-1)^l L^{l-1} M^m L$. Thus

$$\left(\frac{\partial L}{\partial t_{1,k}^*}\right)_- = [(A_{1,k})_+, L]_- + 2(L^k)_-. \tag{4.4}$$

For the cBKP hierarchy, the action of one flow ∂_τ on the eigenfunctions $(\partial_\tau q_i)$ and $(\partial_\tau r_i)$ may be derived from its action on the L if

$$(\partial_\tau L)_- = \sum_{i=1}^{m}((\partial_\tau q_i)\partial^{-1}r_{i,x} - (\partial_\tau r_i)\partial^{-1}q_{i,x} + q_i\partial^{-1}(\partial_\tau r_{i,x}) - r_i\partial^{-1}(\partial_\tau q_{i,x})). \tag{4.5}$$

At the same time, $(\partial_\tau q_{i,x})$ should be consistent with $(\partial_\tau q_i)$, $(\partial_\tau r_{i,x})$ should be consistent with $(\partial_\tau r_i)$. However, the following lemma shows that $\partial_{t^*_{1,k}}$ is not the case due to the form of $(L^k)_-$.

Lemma 4.1. The Lax operator L of the constrained BKP hierarchy given by eq.(4.1) satisfies the relation

$$(L^k)_- = \sum_{i=1}^{m}\sum_{j=0}^{k-1}(L^{k-j-1}(q_i)\partial^{-1}(L^*)^j(r_{i,x}) - L^{k-j-1}(r_i)\partial^{-1}(L^*)^j(q_{i,x})), k = 1,3,5,\cdots \tag{4.6}$$

where

$$L(q_i) = L_+(q_i) + \sum_{j=1}^{m}(q_j\partial_x^{-1}(r_{j,x}q_i) - r_j\partial_x^{-1}(q_{j,x}q_i)).$$

According to the analysis above, we may revise the flows in eq.(4.3) to define a correct additional symmetry flows of the cBKP hierarchy. Similar to the case of cCKP hierarchy, we define a new additional flow

$$\partial_k^* L = [-(A_{1,k})_- + Y_k, L], k = 1,3,5,\cdots, \tag{4.7}$$

where $A_{1,k} = M^m L^l - (-1)^l L^{l-1} M^m L$ is the generator of the additional symmetry of the BKP hierarchy. We shall show in this section that ∂_k^* is a correct additional symmetry flow for the cBKP hierarchy. First of all, by a similar discussion as the case of the cCKP hierarchy, we have the following lemma.

Lemma 4.2. For the cBKP hierarchy, the BKP reduction condition infers a constraint on Y_k as

$$Y_k^* = -\partial Y_k \partial^{-1},\ k = 1,3,5,\cdots \tag{4.8}$$

Thus we can set

$$Y_1 = 0, \tag{4.9}$$

$$Y_k = \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(L^{k-2-j}(q_i)\partial^{-1}(L^*)^j(r_{i,x}) - L^{k-2-j}(r_i)\partial^{-1}(L^*)^j(q_{i,x})), k \geq 3. \tag{4.10}$$

The following lemma shows that $\{Y_k,\ k = 1,3,5,\cdots\}$ satisfy the property we need.

Lemma 4.3.

$$Y_k^* = -\partial Y_k \partial^{-1}. \tag{4.11}$$

Proof. $Y_1^* = Y_1 = 0$ is trivial. For $k = 3,5,7,\cdots$, we have

$$\begin{aligned}
\partial Y_k \partial^{-1} &= \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(\partial L^{k-2-j}(q_i)\partial^{-1}(L^*)^j(r_{i,x})\partial^{-1} - \partial L^{k-2-j}(r_i)\partial^{-1}(L^*)^j(q_{i,x})\partial^{-1}) \\
&= \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))((-1)^{k-2-j}(L^*)^{k-2-j}(q_{i,x})(-1)^j(L^j(r_i)\partial^{-1} - \partial^{-1}L^j(r_i)) \\
&\quad - (-1)^{k-2-j}(-L^*)^{k-2-j}(r_{i,x})(-1)^j(L^j(q_i)\partial^{-1} - \partial^{-1}L^j(q_i)))
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))((L^*)^{k-2-j}(q_{i,x})L^j(r_i)\partial^{-1} - (L^*)^{k-2-j}(q_{i,x})\partial^{-1}L^j(r_i) \\
&- (L^*)^{k-2-j}(r_{i,x})L^j(q_i)\partial^{-1} + (L^*)^{k-2-j}(r_{i,x})\partial^{-1}L^j(q_i)) \\
&= -\sum_{i=1}^{m}\sum_{l=0}^{k-2}(2l-(k-2))(-(L^*)^{l}(r_{i,x})\partial^{-1}L^{k-2-l}(q_i) + (L^*)^{l}(q_{i,x})\partial^{-1}L^{k-2-l}(r_i)) \\
&- \sum_{i=1}^{m}\sum_{j=0}^{k-2}(2j-(k-2))(-1)^{k-2-j}(\partial L^{k-2-j}(q_i)L^j(r_i)\partial^{-1} - \partial L^{k-2-j}(r_i)L^j(q_i)\partial^{-1}) \\
&= -Y_k^*
\end{aligned}
$$

we have used the identity $\partial^{-1}f_x\partial^{-1} = f\partial^{-1} - \partial^{-1}f$ as $f = (L^*)^j(r_{i,x})$ and $f = (L^*)^j(q_{i,x})$ in the derivation. □

In order to get the explicit form of the right hand side of eq.(4.7), the following lemma is necessary.

Lemma 4.4. For the Lax operator L of the cBKP hierarchy and $X = \sum_{k=1}^{l} M_k\partial^{-1}N_k$,

$$
\begin{aligned}
[X, L]_- &= \sum_{k=1}^{l}(-L(M_k)\partial^{-1}N_k + M_k\partial^{-1}L^*(N_k)) \\
&\quad + \sum_{i=1}^{m}(X(q_i)\partial^{-1}r_{i,x} - X(r_i)\partial^{-1}q_{i,x} - q_i\partial^{-1}X^*(r_{i,x}) + r_i\partial^{-1}X^*(q_{i,x}))
\end{aligned}
$$

holds.

Applying the above lemma we conclude that

$$
\begin{aligned}
[Y_k, L]_- = -2(L^k)_- &+ k\sum_{i=1}^{m}(L^{k-1}(q_i)\partial^{-1}r_{i,x} - L^{k-1}(r_i)\partial^{-1}q_{i,x}) \\
&+ k\sum_{i=1}^{m}(q_i\partial^{-1}L^{*(k-1)}(r_{i,x}) - r_i\partial^{-1}L^{*(k-1)}(q_{i,x})) \\
&+ \sum_{i=1}^{m}(Y_k(q_i)\partial^{-1}r_{i,x} - Y_k(r_i)\partial^{-1}q_{i,x}) \\
&- \sum_{i=1}^{m}(-q_i\partial^{-1}Y_k^*(r_{i,x}) + r_i\partial^{-1}Y_k^*(q_{i,x})).
\end{aligned}
$$

Theorem 4.1. For the cBKP hierarchy, the additional flows defined by eq.(4.7) acting on the eigenfunction q_i and r_i are

$$\partial_k^* q_i = kL^{k-1}(q_i) + Y_k(q_i) - (A_{1,k})_+(q_i), \tag{4.12}$$

$$\partial_k^* r_i = kL^{k-1}(r_i) + Y_k(r_i) + (A_{1,k})_+(r_i), \tag{4.13}$$

$$\partial_k^* q_{i,x} = -kL^{*(k-1)}(q_{i,x}) - Y_k^*(q_{i,x}) + (A_{1,k})_+^*(q_{i,x}), \tag{4.14}$$

$$\partial_k^* r_{i,x} = -kL^{*(k-1)}(r_{i,x}) - Y_k^*(r_{i,x}) - (A_{1,k})_+^*(r_{i,x}), k = 1, 3, 5, \cdots . \tag{4.15}$$

Proof. From the revised definition of the additional flows in eq.(4.7) of the cBKP hierarchy, by a short calculation, we have

$$\begin{aligned}
\partial_k^* L_- &= [-(A_{1,k})_- + Y_k, L]_- \\
&= [(A_{1,k})_+, L]_- + 2(L^k)_- + [Y_k, L]_- \\
&= \sum_{i=1}^{m} (kL^{k-1}(q_i) + Y_k(q_i) - (A_{1,k})_+(q_i))\partial^{-1} r_{i,x} \\
&- (kL^{k-1}(r_i) + Y_k(r_i) + (A_{1,k})_+(r_i))\partial^{-1} q_{i,x} \\
&+ q_i \partial^{-1}(-kL^{*(k-1)}(r_{i,x}) - Y_k^*(r_{i,x}) - (A_{1,k})_+^*(r_{i,x})) \\
&- r_i \partial^{-1}(-kL^{*(k-1)}(q_{i,x}) - Y_k^*(q_{i,x}) + (A_{1,k})_+^*(q_{i,x}))
\end{aligned}$$

thus

$$\partial_k^* q_i \quad = kL^{k-1}(q_i) + Y_k(q_i) - (A_{1,k})_+(q_i),$$

the other three identities could be also obtained in the same way. □

Remark 4.1. By a simple calculation, $\partial_k^* q_{i,x}$ and $\partial_k^* r_{i,x}$ are not essential and necessary, because it can be obtained from $\partial_k^* q_i$ and $\partial_k^* r_i$, respectively.

Corollary 4.1. The additional flows act on wave operator S of the constrained BKP hierarchy as

$$\partial_k^* S = (-(A_{1,k})_- + Y_k)S, k = 1, 3, 5, \cdots. \tag{4.16}$$

Theorem 4.2. The additional flows ∂_k^* commute with constrained BKP hierarchy $\dfrac{\partial}{\partial t_{2n+1}}$, i.e.

$$[\partial_k^*, \partial_{t_{2n+1}}] = 0. \tag{4.17}$$

This theorem shows $\partial_k^* (k = 1, 3, 5, \cdots)$ are indeed the additional symmetry flows of the cBKP hierarchy.

Using the identity $Y_k^* = -\partial Y_k \partial^{-1},\ k = 1, 3, 5, \cdots$, we can present the next theorem omitting the proof.

Theorem 4.3. Acting on the space of the wave operator S of the constrained BKP hierarchy, ∂_k^* forms new centerless $W_{1+\infty}^{cB}$-subalgebra of centerless $W_{1+\infty}$.

5. Conclusions and Discussions

To summarize, we have constructed the additional symmetries of the constrained CKP hierarchy in eq.(3.7) and theorem 3.2. The additional flows action on the eigenfunction of the cCKP hierarchy are given in theorem 3.1. Acting on the space of the wave operator S of the cCKP hierarchy, ∂_k^* forms a new centerless $W_{1+\infty}^{cC}$-subalgebra of centerless $W_{1+\infty}$ algebra in theorem 3.3. Similarly, the conclusions for the constrained BKP hierarchy are obtained using the analogous technique in the case of constrained CKP hierarchy. The main results of the cBKP hierarchy are presented in eq.(4.7), theorem 4.1, 4.2 and 4.3. Our results show that the cCKP and cBKP hierarchies have, indeed, some different properties for additional symmetry comparing with the KP, BKP, CKP and constrained KP hierarchies. For example, the definitions of additional symmetry flows for the cCKP and cBKP hierarchies are different from the above hierarchies, the revised operators Y_k of the cCKP and cBKP hierarchies are different from the corresponding operator X_k of the cKP hierarchy given in [25]. Moreover, we also

would like to point out that Y_k of the cCKP and cBKP hierarchies are distinct. We can use the similar technique without essential difficulty to construct additional symmetries of the case as $L^k = B_k + \sum q_i\partial^{-1}r_i + r_i\partial^{-1}q_i, k > 1$, the results will be given in the future.

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